**Calculate the limit**

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$$\lim_{n\to\infty}\frac{1\cdot\ln 1+2\cdot\ln 2+3\cdot\ln 3+\ldots+n\cdot\ln n}{n^2\ln n}.$$

**Solution by Arkady Alt , San Jose, California, USA.**

Let $a_n := \sum_{k=1}^{n} k\ln k, b_n := n^2\ln n.$ Since $\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{(n+1)\ln(n+1)}{(n+1)^2\ln(n+1)-n^2\ln n} =$ $\lim_{n\to\infty}\frac{(n+1)\ln(n+1)}{(n+1)^2\ln n-n^2\ln n+(n+1)^2\ln(1+1/n)} = \lim_{n\to\infty}\frac{\ln(n+1)/\ln n}{(2n+1)\ln n+(n+1)^2\ln(1+1/n)} =$ $\lim_{n\to\infty}\frac{\ln(n+1)/\ln n}{\frac{2n+1}{n+1}+\frac{(n+1)\ln(1+1/n)}{\ln n}} = \frac{1}{2}$ (because $\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} = 1, \lim_{n\to\infty}\frac{2n+1}{n+1} = 2$ and* $\lim_{n\to\infty}\frac{(n+1)\ln(1+1/n)}{\ln n} = 0$) then by Stolz's Theorem $\cdot$

$$\lim_{n\to\infty}\frac{1\cdot\ln 1+2\cdot\ln 2+3\cdot\ln 3+\ldots+n\cdot\ln n}{n^2\ln n} = \frac{1}{2}.$$

* Since $1/(n+1)<\ln(1+1/n) < 1/n$ then $\frac{1}{\ln n} < \frac{(n+1)\ln(1+1/n)}{\ln n} < \frac{n+1}{n\ln n}$ and $\lim_{n\to\infty}\frac{1}{\ln n} = \lim_{n\to\infty}\frac{n+1}{n\ln n} = 0$ by Squeeze Principle implies $\lim_{n\to\infty}\frac{(n+1)\ln(1+1/n)}{\ln n} = 0.$